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Orbital effects of the Magnus force on a spinning spherical satellite in a rarefied atmosphere

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Abstract

The effects of the Magnus force on a spinning sphere in a Keplerian orbit is investigated using perturbation theory. The result is that the plane of the orbit will rotate with the angular velocity $-\frac{1}{4}\alpha_\tau \frac{mn}{\rho_S}\boldsymbol{\omega}$, where α_τ is the accommodation coefficient of tangential momentum, m and n are the mass and number density of the surrounding gas, and where ρ_S and $\boldsymbol{\omega}$ are the mean density and the angular velocity of the sphere. It is shown that under reasonable assumptions, for a spinning satellite in the Earth's atmosphere, this effect is small.

Keywords: Satellite orbit; Kinetic theory; Transversal force; Perturbation theory

1. Introduction

The motion of bodies in Keplerian orbits perturbed by the action of small Atmospheric forces have been the subject of various investigations. Many results on this topic are found in the book by King-Hele, [1]. In [2], the atmospheric influence on small debris particle orbits are studied. Further, the effects of time-varying aerodynamical coefficients on satellite orbits is investigated in [3], where the action of periodic perturbative forces are calculated. In the present investigation we will calculate the effects of another periodic force, namely the Magnus force exerted by a rarefied gas on a spinning satellite in a Keplerian orbit. In doing this, we shall assume that this as well as other atmospheric forces are small and thus only will affect the orbit of the satellite on the timescale of several orbit periods.

Under the assumption that the surrounding fluid is highly rarefied, the direction of the Magnus force acting on a spinning sphere is opposite to what is the case in the continuum region. This may be explained in the following way. We denote the angular velocity vector of the spinning sphere by $\boldsymbol{\omega}$, the velocity vector by $\boldsymbol{v}$, and the outward surface normal of the sphere by $\boldsymbol{n}$. In the continuum region, the fluid velocity at the part of the sphere surface where $\boldsymbol{n} \cdot (\boldsymbol{\omega} \times \boldsymbol{v}) > 0$, the local fluid speed is increased by the viscous action of the sphere surface. Thus, by Bernoulli's theorem, the local pressure is here decreased. On the opposite side of the sphere, the local fluid speed is instead decreased, and thus the local pressure is increased. Therefore a net force acting the sphere is produced, parallel to the negative of the established pressure gradient, that is, in the direction of $\boldsymbol{\omega} \times \boldsymbol{v}$.

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In the case where the surrounding fluid is highly rarefied (where mean free path of the fluid is large compared to the diameter of the sphere), the spinning sphere will hit molecules mainly on the wind-ward side. Upon hitting the sphere surface, these molecules will be deflected by the sphere in the direction of the local surface velocity, that is, in the direction of $\boldsymbol{\omega} \times \boldsymbol{v}$. The recoil action on the spinning sphere will produce a force of the opposite direction, that is, directed parallel to $-\boldsymbol{\omega} \times \boldsymbol{v}$.

In an intermediate region, these two effects will compete, and at a particular point cancel out. The Magnus force acting on a spinning cylinder in this intermediate region is studied by means of Monte-Carlo simulations in [4].

2. Satellite orbit in a rarefied gas

In the absence of a surrounding fluid, only gravity acts upon the satellite, and Newton's second law becomes

$$m_{\mathrm{S}} \frac{\mathrm{d}^2 \boldsymbol{r}}{\mathrm{d}t^2} = -\frac{GMm_{\mathrm{S}}}{r^3} \boldsymbol{r}. \tag{1}$$

Here G is the gravitational constant, M and m_{S} are the masses of the Earth and the satellite and $\boldsymbol{r}$ is the position vector of the satellite with respect to an inertial system with its origin at the center of the Earth, and where $r = |\boldsymbol{r}|$. In this situation, the total translational mechanical energy (not including the energy of the spinning motion of the satellite) E, the (orbital) angular momentum $\boldsymbol{L}$ and the Runge–Lenz vector $\boldsymbol{A}$, of the satellite are conserved. These quantities are defined by

$$E = \frac{m_{\mathrm{S}}}{2} \left| \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \right|^2 - \frac{GMm_{\mathrm{S}}}{r}, \tag{2}$$

$$\boldsymbol{L} = \boldsymbol{r} \times m_{\mathrm{S}} \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}, \tag{3}$$

and

$$\boldsymbol{A} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \times \boldsymbol{L} - \frac{GMm_{\mathrm{S}}\boldsymbol{r}}{r}. \tag{4}$$

The solution for the satellite orbit (i.e. a bounded orbit) is the well known Kepler ellipse [5] given by

$$r(\varphi) = \frac{a(1 - \mathrm{e}^2)}{1 + \mathrm{e}\cos\varphi}, \tag{5}$$

where r and φ are the plane-polar coordinates in a frame with its origin at the centre of the Earth. The parameters a and e are the semi-major axis and the eccentricity of the orbit. These can be related the conserved quantities (3) and (4) according to $\mathrm{e} = A/GMm_{\mathrm{S}}$ and $a(1 - \mathrm{e}^2) = L^2/GMm_S{}^2$. At $\varphi = 0$ the satellite is at Perigee, the point where the orbit assumes its smallest distance to the Earth. The Runge–Lenz vector (4) is directed from the origin towards Perigee, and has the length $A = GMm_{\mathrm{S}}\mathrm{e}$. The time-dependence of φ can be obtained from conservation of orbital angular momentum $m_{\mathrm{S}} r^2 \dot{\varphi} = L$ (here $\dot{\varphi}$ means the time derivative of φ) and the time of a period of the orbit τ obeys Kepler's third law

$$4\pi^2 a^3 = GM\tau^2. \tag{6}$$

Now the influence of the thin surrounding fluid on the orbit of the satellite is taken into account. In doing this we shall make a few assumptions. First we assume that the satellite is spherically symmetric with a radius that is much smaller than the mean free path of the surrounding fluid. Moreover, we also assume that the surrounding fluid is at rest and that its temperature and number density only depend on the distance to the center of the Earth $r = |\boldsymbol{r}|$. Thus the temperature T and number density n to be used at a specific height are average values of these quantities at the radius r. Further, we assume that the surrounding fluid can be approximated as a neutral, highly rarefied gas in its interaction with the satellite's surface. For the temperature of the surface of the satellite we assume that it is constant over the body, but may depend on the height of the satellite, and implicitly (via r) also on the speed of the satellite. In reality, this temperature is determined by the energy flux through the surface from the interaction with the molecules of the surrounding fluid and also by thermal radiation.

We observe that for these assumptions to hold, for a satellite of a small enough size, the region under consideration is the thermosphere and possibly above, and that the speed of the satellite here exceeds the thermal speed of the surrounding gas [6]. Thus we may calculate the forces and torques exerted on the satellite by the surrounding gas in the high-speed limit. If we denote the temperature of the satellite's surface by T_{w}, the drag force becomes [7]

$$\boldsymbol{F}_{\mathrm{drag}} = -\frac{\pi R^2}{4} mn \sqrt{\frac{2k_B T}{m}} \left[(1+\alpha_\tau) \sqrt{\frac{m}{2k_B T}} \left| \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \right| + \frac{2\sqrt{\pi}}{3} \alpha_\tau \sqrt{\frac{T_{\mathrm{w}}}{T}} \right] \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}. \tag{7}$$

In this expression, R is the radius of the satellite, α_τ is the accommodation coefficient of tangential momentum, measuring the relative amount of the gas molecules incident on the surface of the satellite to be reflected diffusely. Further, m is the mass of a gas molecule in the thermosphere and n is the number density of the molecules. Since the satellite is spinning a Magnus force is also acting on it. In the present situation where the diameter of the sphere is much smaller than the mean free path this force has the opposite sign compared to the corresponding force appearing in the continuum limit [8], and is given by

$$\boldsymbol{F}_{\mathrm{Magnus}} = -\alpha_\tau \frac{2\pi R^3}{3} mn\, \boldsymbol{\omega} \times \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}. \tag{8}$$

Here $\boldsymbol{\omega}$ is the angular velocity vector of the sphere. Further, a torque exerted by the thermosphere is damping the angular velocity of the spinning sphere. This torque (with respect to the center of the sphere) is in the present case given by [9]

$$-\alpha_\tau \frac{3\pi R^4}{4} mn \left[\left| \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \right| \boldsymbol{\omega} - \frac{4}{9} \left| \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \right|^{-1} \left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \cdot \boldsymbol{\omega} \right) \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \right]. \tag{9}$$

3. Motion on two time scales

There are two time scales present in the motion of the satellite: the period time of the unperturbed Keplerian orbit τ, given by (6), and also the typical time by which the velocity and angular velocity decay under the action of the drag force (7) and the torque (9). Using the expression for the drag force this decay time becomes

$$\frac{16}{3} \frac{\rho_S}{mn} R \left[(1+\alpha_\tau) \left| \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \right| + \frac{2}{3} \sqrt{\frac{2\pi k_B T_{\mathrm{w}}}{m}} \right]^{-1}, \tag{10}$$

where ρ_S is the mean density of the satellite. Far above the Earth's surface, the drag force is small enough for this decay time to be much larger than the orbit period time of the corresponding unperturbed orbit, τ. Due to the action of the drag force a satellite in this region will slowly descend through a spiral shaped orbit, and as the height decreases the number density of the surrounding gas will increase. Thus, at a certain height, the size of the satellite will no longer be small compared to the mean free path. We shall refer to this height as the transitional height. For a spherical satellite of radius 1 meter, this transitional height is approximately 130 km above the Earth's surface. Below that height, the forces (7) and (8), and the torque (9) will no longer hold.[1]

We will in the following use the values of the number density and the temperature at the transitional height, denoted by n_{t} and T_{t}, as the scales of the number density and temperature. Inserted into the decay time of the velocity given by (10), this decay time will probably assume its minimum value. We introduce a non-dimensional number density and a non-dimensional gas and satellite surface temperature according to

$$n = n_{\mathrm{t}} n^*(r), \quad T = T_{\mathrm{t}} T^*(r) \quad \text{and} \quad T_{\mathrm{w}} = T_{\mathrm{t}} T_{\mathrm{w}}^*(r). \tag{11}$$

Next, we introduce the non-dimensional variables

$$t^* = 2\pi t/\tau_{\mathrm{i}}, \quad \boldsymbol{r}^* = \boldsymbol{r}/a_{\mathrm{i}} \quad \text{and} \quad \boldsymbol{\omega}^* = \boldsymbol{\omega}/\omega_{\mathrm{i}}.$$

Here, a_{i} is the semi-major axis of the initial unperturbed Keplerian orbit, and τ_{i} is the corresponding period time given by Kepler's Law. ω_{i} is the absolute value of the initial angular velocity. With this scaling the non-dimensional Runge–

[1] It is likely that a somewhat similar set of forces and torques will act on the spinning satellite below this height [11], and the resulting motion of the satellite may possibly be analysed in a way similar to the present investigation. This requires however that the decay time will remain large compared to the period time of the orbit, which is the basis upon which the analysis below rests.

Lenz vector $\boldsymbol{A}^*$ fulfils $\boldsymbol{A}^* = \boldsymbol{A}/GMm_{\mathrm{S}}$, and thus $|\boldsymbol{A}^*| = \mathrm{e}$, the eccentricity. Further, we define the small parameter ε as the quotient between the period time τ_{i} and the damping time (10) according to

$$\varepsilon = \frac{3}{16}\frac{mn_{\mathrm{t}}}{\rho_S}\frac{a_{\mathrm{i}}}{R}. \tag{12}$$

Here we have assumed that the first term within the square brackets of (10) dominates. Typically, using $mn_{\mathrm{t}} \sim 10^{-8}$ kg m^{-3} at the height of 130 km above the Earth's surface [10], and using a satellite density corresponding to a metal, $\varepsilon \sim 10^{-8}$–10^{-7}. The corresponding non-dimensional equation of motion is then given by (the *-superscript will be dropped in what follows):

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -\frac{\boldsymbol{r}}{r^3} - \varepsilon\, n(r)\left[\left(k_1\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right| + k_2\sqrt{T_{\mathrm{w}}(r)}\right)\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + k_3\boldsymbol{\omega}\times\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right]. \tag{13}$$

For convenience, the coefficients k_1, k_2 and k_3 have been introduced. These are given by

$$k_1 = (1+\alpha_\tau), \qquad k_2 = \frac{\alpha_\tau}{3}\frac{\sqrt{k_B T_{\mathrm{t}}/2\pi m}}{2\pi a_{\mathrm{i}}/\tau_{\mathrm{i}}}, \qquad k_3 = \alpha_\tau \frac{8}{3}\frac{R\omega_{\mathrm{i}}}{2\pi a_{\mathrm{i}}/\tau_{\mathrm{i}}}. \tag{14}$$

It can be noted that the coefficients k_2 and k_3 are proportional to the quotients between the thermal speed and the rotational speed, respectively, of the unperturbed satellite motion and the initial translational speed. The non-dimensional Euler equation of the spin angular momentum is then given by

$$\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} = -\varepsilon n(r) k_4\left[\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|\boldsymbol{\omega} - \frac{4}{9}\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|^{-1}\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\cdot\boldsymbol{\omega}\right)\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right], \tag{15}$$

where the constant k_4 is given by

$$k_4 = \alpha_\tau \frac{2}{15\pi}, \tag{16}$$

and where for simplicity we have used the moment of inertia $2m_{\mathrm{S}}R^2/5$ of a homogeneous sphere. At $t = 0$, $\boldsymbol{\omega}$ is the unit vector parallel to the initial angular velocity.

Equations for the time evolution of E, $\boldsymbol{L}$ and $\boldsymbol{A}$ can be obtained from the equation of motion (13) and one gets

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\varepsilon n(r)\left(k_1\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|^3 + k_2\sqrt{T_{\mathrm{w}}(r)}\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|^2\right), \tag{17}$$

$$\frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = -\varepsilon n(r)\left(k_1\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right| + k_2\sqrt{T_{\mathrm{w}}(r)}\right)\boldsymbol{L} + k_3\left[\left(\boldsymbol{r}\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)\boldsymbol{\omega} - (\boldsymbol{r}\cdot\boldsymbol{\omega})\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right], \tag{18}$$

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = -\varepsilon n(r)\bigg\{&\left(2k_1\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right| + k_2\sqrt{T_{\mathrm{w}}(r)}\right)\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\times\boldsymbol{L} \\ &+ k_3\left(\boldsymbol{r}\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\times\boldsymbol{\omega} + k_3\left[\boldsymbol{r}\cdot\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\times\boldsymbol{\omega}\right)\right]\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\bigg\}.\end{aligned} \tag{19}$$

4. Solution with slowly varying parameters

We now assume that the variations of the number density of the surrounding gas $n(r)$ during one period is negligible. This is a valid approximation if the eccentricity of the orbit is not too large and if $n(r)$ is not a too rapidly changing function of r. When this is done we make the anzats that the forces and torques that stem from the surrounding gas will influence the orbit only on a time scale defined by $\tilde{t} = \varepsilon t$, much slower than the time scale t of the unperturbed orbit. Technically this means that the time-derivative is split into two parts according to

$$\frac{\mathrm{d}}{\mathrm{d}t} \to \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial\tilde{t}}.$$

Accordingly, the energy, the orbital angular momentum and the Runge–Lenz vector are then regarded as functions of both t and $\tilde{t}$, and are expanded in the small parameter ε according to $E = E_0(t,\tilde{t}) + \varepsilon E_1(t,\tilde{t}) + \cdots$, $\boldsymbol{L} = \boldsymbol{L}_0(t,\tilde{t}) + \varepsilon\boldsymbol{L}_1(t,\tilde{t}) + \cdots$ and $\boldsymbol{A} = \boldsymbol{A}_0(t,\tilde{t}) + \varepsilon\boldsymbol{A}_1(t,\tilde{t}) + \cdots$. The position vector $\boldsymbol{r}$ and the rotation $\boldsymbol{\omega}$ are expanded in the same way. Since the quantities E_0, $\boldsymbol{L}_0$, $\boldsymbol{A}_0$ and $\boldsymbol{\omega}_0$ do not change during a period, Eqs. (17), (18), (19) and (15) to order zero in ε give $E_0 = E_0(\tilde{t})$, $\boldsymbol{L}_0 = \boldsymbol{L}_0(\tilde{t})$, $\boldsymbol{A}_0 = \boldsymbol{A}_0(\tilde{t})$ and $\boldsymbol{\omega}_0 = \boldsymbol{\omega}_0(\tilde{t})$. We will not pursue the slow-time evolution of $\boldsymbol{\omega}_0$ further.

To first order in ε, the equation for the energy E becomes

$$\frac{\partial E_1}{\partial t} = -\frac{\partial E_0}{\partial \tilde{t}} - n(r_0)\left(k_1\left|\frac{\partial \boldsymbol{r}_0}{\partial t}\right|^3 + k_2\sqrt{T_{\mathrm{w}}(r_0)}\left|\frac{\partial \boldsymbol{r}_0}{\partial t}\right|^2\right). \tag{20}$$

Now the second term on the right-hand side is a periodic function in the fast time variable t. This means that it can be expanded in a Fourier series. The constant term in this expansion, which is the period average of the expanded function, will cause a linear growth of E_1 with t and will thus give a term in E proportional to ϵt which will cause a breakdown of the perturbation for times of the order of the damping time. The idea is now to avoid this by requiring that the slow time derivative of E_0 in the left-hand side of (20) cancels this constant term [12]. This gives, if the average value over a period of a quantity f is denoted by $\langle f\rangle$,

$$\frac{\partial E_0}{\partial \tilde{t}} = -n(r_0)\left[k_1\left\langle\left|\frac{\partial \boldsymbol{r}_0}{\partial t}\right|^3\right\rangle + k_2\left\langle\sqrt{T_{\mathrm{w}}(r_0)}\left|\frac{\partial \boldsymbol{r}_0}{\partial t}\right|^2\right\rangle\right], \tag{21}$$

where we have used the fact that n does not change over a period. This equation describes a decrease of the energy of the orbit with time. This is due to the action of the friction force. The Virial theorem states for an unperturbed Keplerian orbit, with kinetic energy T and total energy E, that $\langle T\rangle = -\langle V\rangle/2$, and thus, with the present scaling,

$$E = -\frac{1}{2}\left\langle\frac{1}{|\boldsymbol{r}_0|}\right\rangle.$$

It is easy to see that Eq. (21) implies that $\langle|\boldsymbol{r}_0|\rangle$ decreases, as expected. For the special case of an initially circular orbit, we have

$$\frac{\partial r_0}{\partial \tilde{t}} = -n(r_0)\left[k_1\sqrt{r_0} + k_2\sqrt{T_{\mathrm{w}}(r_0)}\,r_0\right]. \tag{22}$$

In the same way, we get from (18) the slow time derivative of $\boldsymbol{L}_0$

$$\frac{\partial \boldsymbol{L}_0}{\partial \tilde{t}} = -n(r_0)\left\langle\left(k_1\left|\frac{\partial \boldsymbol{r}_0}{\partial t}\right| + k_2\sqrt{T_{\mathrm{w}}(r_0)}\right)\boldsymbol{L}_0\right\rangle - n(r_0)k_3\left[\left\langle\left(\boldsymbol{r}_0\cdot\frac{\partial \boldsymbol{r}_0}{\partial t}\right)\boldsymbol{\omega}_0\right\rangle - \left\langle(\boldsymbol{r}_0\cdot\boldsymbol{\omega}_0)\frac{\partial \boldsymbol{r}_0}{\partial t}\right\rangle\right]. \tag{23}$$

This equation can be simplified if one observes that exact derivatives of periodic quantities with respect to t vanish when averaged over a period. That is, if f is a periodic quantity we have

$$\left\langle\frac{\partial f}{\partial t}\right\rangle = \frac{1}{T}\int_0^T \frac{\partial f}{\partial t}\,\mathrm{d}t = \frac{1}{T}\left[f(T) - f(0)\right] = 0.$$

The first term within the square brackets of (23) can be written as $\langle\frac{\partial}{\partial t}(|\boldsymbol{r}_0|^2\boldsymbol{\omega}_0/2)\rangle$ and thus vanishes. The second term within the square brackets can be rewritten as (remember that $\boldsymbol{\omega}_0$ is independent of the fast time t)

$$(\boldsymbol{r}_0\cdot\boldsymbol{\omega}_0)\frac{\partial \boldsymbol{r}_0}{\partial t} = \frac{\partial}{\partial t}\left[\frac{(\boldsymbol{\omega}_0\cdot\boldsymbol{r}_0)\boldsymbol{r}_0}{2}\right] - \frac{1}{2}\boldsymbol{\omega}_0\times\boldsymbol{L}_0.$$

When taking the average over a period the exact time derivative vanishes and the following equation for $\boldsymbol{L}_0$ results:

$$\frac{\partial \boldsymbol{L}_0}{\partial \tilde{t}} = -n(r_0)\left\langle\left(k_1\left|\frac{\partial \boldsymbol{r}_0}{\partial t}\right| + k_2\sqrt{T_{\mathrm{w}}(r_0)}\right)\right\rangle\boldsymbol{L}_0 - \frac{1}{2}n(r_0)k_3\,\boldsymbol{\omega}_0\times\boldsymbol{L}_0, \tag{24}$$

where the first part reflects the effect of the atmospheric drag on the contraction of the angular momentum, while the second part reflects the rotation of the angular momentum vector by the Magnus effect. The slow-time evolution of the absolute value of $\boldsymbol{L}_0$, denoted here by L_0, can be obtained by taking the dot product of Eq. (24) with $\boldsymbol{L}$, and one gets

$$\frac{\partial L_0}{\partial \tilde{t}} = -n(r_0)\left\langle\left(k_1\left|\frac{\partial \boldsymbol{r}_0}{\partial t}\right| + k_2\sqrt{T_{\mathrm{w}}(r_0)}\right)\right\rangle L_0. \tag{25}$$

It is clear that L_0 decreases with time, at a rate that is determined by the mean speed, the mean value of the surface temperature and by the number density of the surrounding gas. Further, if we denote the unit vector normal to the plane of the orbit by $\boldsymbol{e}_\mathrm{n}$ we can write $\boldsymbol{L}_0 = L_0\boldsymbol{e}_\mathrm{n}$ and thus

$$\frac{\partial \boldsymbol{L}_0}{\partial \tilde{t}} = \frac{\partial L_0}{\partial \tilde{t}}\boldsymbol{e}_\mathrm{n} + L_0\frac{\partial \boldsymbol{e}_\mathrm{n}}{\partial \tilde{t}}.$$

This is substituted into (24), and using (25) we get an equation for the normal vector $\boldsymbol{e}_\mathrm{n}$ of the plane of the orbit:

$$\frac{\partial \boldsymbol{e}_\mathrm{n}}{\partial \tilde{t}} = -\frac{1}{2}n(r_0)k_3\boldsymbol{\omega}_0 \times \boldsymbol{e}_\mathrm{n}. \tag{26}$$

Thus, if $\boldsymbol{\omega}_0$ has a component in the plane of the orbit, the plane of the orbit will rotate slowly with an angular velocity parallel to the negative of this component of $\boldsymbol{\omega}_0$. In dimensional units, the rotation of the normal vector is given by

$$\frac{\mathrm{d}\boldsymbol{e}_\mathrm{n}}{\mathrm{d}t} = -\alpha_\tau\frac{1}{4}\frac{mn}{\rho_S}\,\boldsymbol{\omega}_0 \times \boldsymbol{e}_\mathrm{n}. \tag{27}$$

The speed of this rotation is also determined by the number density of the surrounding gas. The friction force does not enter into the equation of $\boldsymbol{e}_\mathrm{n}$. It is clear from (27) that the largest orbital plane changes occur when the spin is aligned with the orbital plane.

The Runge–Lenz vector is governed by the equation

$$\begin{aligned}\frac{\partial \boldsymbol{A}_0}{\partial \tilde{t}} = {}& -2n(r_0)\left\langle\left(k_1\left|\frac{\partial \boldsymbol{r}_0}{\partial t}\right| + k_2\sqrt{T_\mathrm{w}(r_0)}\right)\frac{\partial \boldsymbol{r}_0}{\partial t}\times \boldsymbol{L}_0\right\rangle \\ & -n(r_0)k_3\left\langle\left(\boldsymbol{r}_0\cdot\frac{\partial \boldsymbol{r}_0}{\partial t}\right)\frac{\partial \boldsymbol{r}_0}{\partial t}\times\boldsymbol{\omega}_0\right\rangle - n(r_0)k_3\left\langle\left[\boldsymbol{r}_0\cdot\left(\frac{\partial \boldsymbol{r}_0}{\partial t}\times\boldsymbol{\omega}_0\right)\right]\frac{\partial \boldsymbol{r}_0}{\partial t}\right\rangle.\end{aligned} \tag{28}$$

The last time average term vanishes, since

$$\boldsymbol{r}_0\cdot\left(\frac{\partial \boldsymbol{r}_0}{\partial t}\times\boldsymbol{\omega}_0\right) = \boldsymbol{\omega}_0\cdot\left(\boldsymbol{r}_0\times\frac{\partial \boldsymbol{r}_0}{\partial t}\right) = \boldsymbol{\omega}_0\cdot\boldsymbol{L}_0,$$

is independent of t and thus the term is an average of an exact time derivative. If f and g are periodic quantities, we must have, from integration by parts,

$$\left\langle\frac{\partial f}{\partial t}g\right\rangle = 0 - \left\langle\frac{\partial g}{\partial t}f\right\rangle.$$

Therefore the second term in (28) may be rewritten according to (remember that $\boldsymbol{\omega}$ is independent of the fast time t)

$$\left\langle\left(\boldsymbol{r}_0\cdot\frac{\partial \boldsymbol{r}_0}{\partial t}\right)\frac{\partial \boldsymbol{r}_0}{\partial t}\times\boldsymbol{\omega}_0\right\rangle = \left\langle\frac{\partial}{\partial t}\left(\frac{r_0^2}{2}\right)\frac{\partial \boldsymbol{r}_0}{\partial t}\times\boldsymbol{\omega}_0\right\rangle = -\left\langle\frac{r_0^2}{2}\left(\frac{\partial^2 \boldsymbol{r}_0}{\partial t^2}\times\boldsymbol{\omega}_0\right)\right\rangle.$$

Using the equation of motion (13) to order zero in ε,

$$\frac{\partial^2 \boldsymbol{r}_0}{\partial t^2} = -\frac{\boldsymbol{r}_0}{r_0^3},$$

we get

$$-\left\langle\frac{r_0^2}{2}\left(\frac{\partial^2 \boldsymbol{r}_0}{\partial t^2}\times\boldsymbol{\omega}_0\right)\right\rangle = \frac{1}{2}\boldsymbol{\omega}_0\times\left\langle-\frac{\boldsymbol{r}_0}{r_0}\right\rangle.$$

We also know that

$$\boldsymbol{A}_0 = \langle\boldsymbol{A}_0\rangle = \left\langle\frac{\partial \boldsymbol{r}_0}{\partial t}\times\boldsymbol{L}_0\right\rangle - \left\langle\frac{\boldsymbol{r}_0}{r_0}\right\rangle.$$

As $\boldsymbol{L}_0$ is independent of t the first term here on the right hand side vanishes. Thus we finally get for the second term in (28)

$$\left\langle\left(\boldsymbol{r}_0\cdot\frac{\partial \boldsymbol{r}_0}{\partial t}\right)\frac{\partial \boldsymbol{r}_0}{\partial t}\times\boldsymbol{\omega}_0\right\rangle = \frac{1}{2}\,\boldsymbol{\omega}_0\times\boldsymbol{A}_0,$$

and the equation for the Runge–Lenz vector becomes

$$\frac{\partial \boldsymbol{A}_0}{\partial \tilde{t}} = -2n(r_0)\left\langle \left(k_1\left|\frac{\partial \boldsymbol{r}_0}{\partial t}\right| + k_2\sqrt{T_\mathrm{w}(r_0)}\right)\left(\boldsymbol{A}_0 + \frac{\boldsymbol{r}_0}{r_0}\right)\right\rangle - \frac{1}{2}n(r_0)k_3\,\boldsymbol{\omega}_0 \times \boldsymbol{A}_0. \tag{29}$$

In this expression, we have used the definition of $\boldsymbol{A}$ in the first term on the right-hand side. Taking the dot product of this equation with $\boldsymbol{A}_0$ one gets (here $|\boldsymbol{A}_0|$ is denoted by A_0)

$$A_0\frac{\partial A_0}{\partial \tilde{t}} = -2n(r_0)\left\langle \left(k_1\left|\frac{\partial \boldsymbol{r}_0}{\partial t}\right| + k_2\sqrt{T_\mathrm{w}(r_0)}\right)\left(\boldsymbol{A}_0 + \frac{\boldsymbol{r}_0}{r_0}\right)\cdot \boldsymbol{A}_0\right\rangle. \tag{30}$$

We now define a vector of unit length $\boldsymbol{e}_\mathrm{p}$ pointing to the Perigee by $\boldsymbol{A}_0 = A_0\,\boldsymbol{e}_\mathrm{p}$. Using $\boldsymbol{r}_0 \cdot \boldsymbol{A}_0 = r_0 A_0 \cos\varphi_0$ and $\boldsymbol{A}_0 = \langle -\boldsymbol{r}_0/r_0\rangle$ we get

$$A_0 = \boldsymbol{e}_\mathrm{p}\cdot \boldsymbol{A}_0 = \boldsymbol{e}_\mathrm{p}\cdot\left\langle -\frac{\boldsymbol{r}_0}{r_0}\right\rangle = -\langle\cos\varphi_0\rangle.$$

Thus we get from dividing (30) with A_0:

$$\frac{\partial A_0}{\partial \tilde{t}} = -2n(r_0)\left\langle \left(k_1\left|\frac{\partial \boldsymbol{r}_0}{\partial t}\right| + k_2\sqrt{T_\mathrm{w}(r_0)}\right)\big(-\langle\cos\varphi_0\rangle + \cos\varphi_0\big)\right\rangle.$$

It is obvious that $\langle -\langle\cos\varphi_0\rangle + \cos\varphi_0\rangle = 0$. Since the speed is larger when the orbit is close to the Earth (that is, since the speed is a positive, growing function of $\cos\varphi_0$), we must have

$$\left\langle \left|\frac{\partial \boldsymbol{r}_0}{\partial t}\right|\big(-\langle\cos\varphi_0\rangle + \cos\varphi_0\big)\right\rangle \geqslant 0.$$

It is reasonable to assume that T_w is a decreasing function of r_0. This is due to the fact that as r_0 increases, the speed decreases, and thus the surface temperature T_w will decrease due to the decreased amount of translational energy brought to the surface by the molecules of the thermosphere. Under this assumption we get

$$\left\langle \sqrt{T_\mathrm{w}(r_0)}\big(-\langle\cos\varphi_0\rangle + \cos\varphi_0\big)\right\rangle \geqslant 0,$$

and thus

$$\frac{\partial A_0}{\partial \tilde{t}} = -2n(r_0)\left\langle \left(k_1\left|\frac{\partial \boldsymbol{r}_0}{\partial t}\right| + k_2\sqrt{T_\mathrm{w}(r_0)}\right)\big(-\langle\cos\varphi_0\rangle + \cos\varphi_0\big)\right\rangle \leqslant 0, \tag{31}$$

where the equality holds when the orbit is a circle. This equation thus states that A_0, or the eccentricity, decreases with time: An initially elliptic orbit will become more and more circular. A Taylor expansion of the right-hand side for the case of a small eccentricity orbit yields to first order in the eccentricity

$$\frac{\partial A_0}{\partial \tilde{t}} = -n(r_0)\left[k_1 - k_2\frac{T'_\mathrm{w}(\langle r_0\rangle)\langle r_0\rangle}{2\sqrt{T_\mathrm{w}(\langle r_0\rangle)}}\right]A_0. \tag{32}$$

Again, if the surface temperature decreases with height, the derivative $T'_\mathrm{w}(\langle r_0\rangle)$ is negative. In this case, (32) predicts a decay of the eccentricity, and a circular orbit stays a circle. This agrees with the results obtained in [2]. It is here consistent to use (22) as describing the decay of the mean orbit radius $\langle r_0\rangle$ and rewrite Eq. (32) for $A_0 = A_0(\tilde{t})$ as an equation for $A_0 = A_0(\langle r_0\rangle)$, which yields

$$\frac{\mathrm{d}\ln(A_0)}{\mathrm{d}\langle r_0\rangle} = \frac{2k_1\sqrt{T_\mathrm{w}(\langle r_0\rangle)} - k_2 T'_\mathrm{w}(\langle r_0\rangle)\langle r_0\rangle}{2k_1\sqrt{T_\mathrm{w}(\langle r_0\rangle)\langle r_0\rangle} + k_2 T_\mathrm{w}(\langle r_0\rangle)}. \tag{33}$$

This equation can now be integrated, and if we denote the initial mean orbit radius by $\langle r_0\rangle_\mathrm{i}$ we get

$$A_0(\langle r_0\rangle) = A_0\big(\langle r_0\rangle_\mathrm{i}\big)\exp\left\{\int_{\langle r_0\rangle_\mathrm{i}}^{\langle r_0\rangle} \frac{2k_1\sqrt{T_\mathrm{w}(s)} - k_2T'_\mathrm{w}(s)s}{2k_1\sqrt{T_\mathrm{w}(s)s} + k_2T_\mathrm{w}(s)}\,\mathrm{d}s\right\}. \tag{34}$$

An equation for the Perigee vector $\boldsymbol{e}_\mathrm{p}$ can be obtained from Eq. (29) in much the same way as the equation for the normal vector $\boldsymbol{e}_\mathrm{n}$ was obtained from (24). Using here that the term perpendicular to $\boldsymbol{e}_\mathrm{p}$ and $\boldsymbol{e}_\mathrm{n}$ appearing in the first term on the right-hand side of (29) is proportional to

$$\left\langle \left[k_1 \left| \frac{\partial \boldsymbol{r}_0}{\partial t} \right| + k_2 \sqrt{T_\mathrm{w}(r_0)} \right] \sin\varphi_0 \right\rangle$$

and thus vanishes as $\sin\varphi_0$ is odd in φ_0 but the other terms are even, we get

$$\frac{\partial \boldsymbol{e}_\mathrm{p}}{\partial \tilde{t}} = -\frac{1}{2} n(r_0) k_3 \boldsymbol{\omega}_0 \times \boldsymbol{e}_\mathrm{p}. \tag{35}$$

For a circular orbit the rotation of Perigee is not well defined, and the effect prescribed by (35) is collapsed into a slowly varying change in the phase of the orbit.

An estimate of the arc length Δs the orbital plane has rotated can now be obtained. For simplicity we will focus on a circular orbit of radius r. If we denote the angular velocity vector of the rotation of the plane of the orbit by $\boldsymbol{\gamma}$, we have from (27) that

$$\boldsymbol{\gamma} = -\alpha_\tau \frac{1}{4} \frac{mn}{\rho_S} \boldsymbol{\omega}_0.$$

If $\tilde{r}$ is the average radius of the orbit during the time interval Δt and if $\tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma}|_{t=0}$ we may approximate the arc length Δs according to

$$\Delta s \sim |\tilde{\boldsymbol{\gamma}}| \tilde{r} \Delta t.$$

If we now denote the orbit contraction during Δt by Δr, we have

$$\frac{\Delta s}{\Delta r} \sim \frac{|\tilde{\boldsymbol{\gamma}}| \tilde{r}}{\Delta r / \Delta t}.$$

The quotient $\Delta r/\Delta t$ is now approximated with

$$\left. \frac{\mathrm{d}r_0}{\mathrm{d}t} \right|_{r=\tilde{r}},$$

which becomes, through (22) in dimensional units,

$$\left. \frac{\mathrm{d}r_0}{\mathrm{d}t} \right|_{r=\tilde{r}} = -\frac{3}{4\pi} \frac{mn}{\rho_S} \frac{1}{R} \left[\pi(1+\alpha_\tau) \frac{\tilde{r}^2}{\tau_\mathrm{i}} + \frac{\alpha_\tau}{3} \sqrt{\frac{2k_B T_\mathrm{w}(\tilde{r})}{m}} \tilde{r} \right],$$

and we get

$$\frac{\Delta s}{\Delta r} \sim \frac{-\alpha_\tau \frac{1}{4} \frac{mn}{\rho_S} |\boldsymbol{\omega}_0| \tilde{r}}{-\frac{3}{4\pi} \frac{mn}{\rho_S} \frac{1}{R} \left[\pi(1+\alpha_\tau) \frac{\tilde{r}^2}{\tau_\mathrm{i}} + \frac{\alpha_\tau}{3} \sqrt{\frac{2k_B T_\mathrm{w}(\tilde{r})}{m}} \tilde{r} \right]}.$$

Using the rough estimate $\tilde{r} \sim r_E$ (the radius of the Earth) we have

$$\frac{\Delta s}{\Delta r} \sim \frac{|\boldsymbol{\omega}_0| R}{r_E/\tau_\mathrm{i} + \sqrt{2k_B T_\mathrm{w}(r_E)/m}}. \tag{36}$$

In this expression, the number density drops out. As an example, consider a satellite in an orbit at the initial height 300 km above the Earth, rotating with 1 revolution per second with an angular velocity $\boldsymbol{\omega}_0$ parallel with the plane of the orbit. If the satellite has the radius 1 m this means that the transitional height is 130 km, where the mean free path is around 10 m. The speed of the satellite is here much larger than the thermal speed of the surrounding gas. The temperature of the surface of the satellite is larger than the temperature of the surrounding gas, but since the thermal speed is a slowly growing function of the temperature, we conclude that in the present situation the second term in the denominator of (36) can be neglected. Then we find that during the contraction of the orbit from 300 km to 130 km the angle the orbit has rotated an angle 10^{-4} radians, which corresponds to the orbit turning $\Delta s \sim 20$ m. Thus in this particular context the effect is small.

5. Conclusion

It has been shown that the action of a thin atmosphere makes the orbit of a spinning spherical satellite to slowly rotate, apart from slowly contracting the orbit radius. The angular velocity of this rotation was found to be parallel to the negative of the angular velocity of the spinning satellite, and is given by

$$-\alpha_\tau \frac{1}{4}\frac{mn}{\rho S}\boldsymbol{\omega}_0.$$

This means that the component of $\boldsymbol{\omega}_0$ parallel to the plane of the orbit will turn the plane of the orbit, whereas the component of $\boldsymbol{\omega}_0$ orthogonal to the plane of the orbit will rotate the Perigee of the orbit. Under reasonable assumptions, the effect on a spinning satellite in the Earth's atmosphere is small. The eccentricity of the orbit is found to decrease under reasonable assumption for the temperature of the surface of the satellite. This is in agreement with previously obtained results.

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